GEOMETRY.

433. Proposed by R. P. BAKER, University of Iowa.

A transformation of the plane keeping the radius of curvature of all curves invariant is either (1) a real or imaginary motion or reflexion, or (2) not a point transformation.

Note.—By mistake this problem was originally credited to W. H. BUSSEY, instead of to R. P. BAKER, who is the rightful Proposer.—EDITORS.

SOLUTION BY ELIJAH SWIFT, University of Vermont.

Suppose that this is a point transformation transforming points in plane I to points in plane II. If one configuration is transformed into another, the second is called the *image* of the first. The theorem is not necessarily true if only a finite number of points or even a one-dimensional set be transformed, and I shall assume that the transformation is defined over a two-dimensional region of I possessing at least one interior point and over a similar region of II, and that it is one-to-one in these regions. Now the image of a straight line is a straight line and that of a circle an equal circle. Hence the transformation satisfies the conditions given in an article of mine in the *Bulletin of the American Mathematical Society* (2d series, vol. x, 1903–1904, pp. 247 ff.), and hence must be a collineation. If it is to transform a circle into a circle, it must leave the circular points at infinity invariant (or interchange them) and hence is a finite collineation of the form $x' = a_1x + b_1y + c_1$, $y' = a_2x + b_2y + c_2$. Omitting the c's, which corresponds to making a translation, and writing the condition that $x^2 + y^2 = r^2$ is carried into itself, we obtain the equations $a_1^2 + a_2^2 = b_1^2 + b_2^2 = 1$, $a_1b_1 + a_2b_2 = 0$. If we put $a_1 = \cos\alpha$, $a_2 = \sin\alpha$, then we find that $b_1 = \sin\alpha$, $b_2 = -\cos\alpha$, or $b_1 = -\sin\alpha$, $b_2 = \cos\alpha$. Our collineation is then either of the form

$$\left.\begin{aligned} x' &= \cos\alpha x + \sin\alpha y + c_1 \\ y' &= \sin\alpha x - \cos\alpha y + c_2 \end{aligned}\right\},$$

which corresponds to (1) a rotation through the angle α, (2) a translation through a distance c_1 along the x-axis, and a distance $-c_2$ long the y-axis, and (3) a reflexion in the x-axis; or else of the form

$$\left.\begin{aligned} x' &= \cos\alpha x + \sin\alpha y + c_1 \\ y' &= -\sin\alpha x + \cos\alpha y + c_2 \end{aligned}\right\},$$

which corresponds to a rotation followed by a translation.

447. Proposed by HORACE OLSON, Chicago, Ill.

Given the edge of a regular tetrahedron, find the radius of the circumscribed sphere.

SOLUTION BY MRS. ELIZABETH BROWN DAVIS, U. S. Naval Observatory.

Let A-BCD be a regular tetrahedron, whose edge is a. Let the radius of the circumsphere be R. In the $\triangle\ BCD$, let the medians meet in O. Then $BC = a$; $BB' = \dfrac{a\sqrt{3}}{2}$; $BO = \dfrac{2}{3} \times \dfrac{a\sqrt{3}}{2} = \dfrac{a\sqrt{3}}{3}$. Since $\triangle\ BCD$ is equilateral, O is

equidistant from BC, CD, and BD, and hence equidistant from the planes A-BC, A-CD, and A-BD. Hence O lies on the line of intersection of the three planes bisecting the dihedral angles C-AB-D, B-AD-C, and D-AC-B.

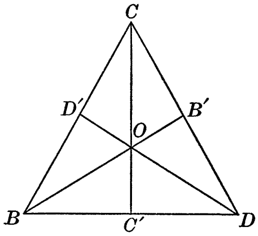

FIG. 1.

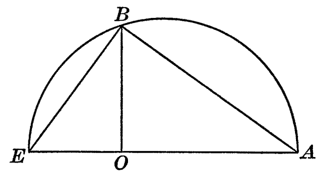

FIG. 2.

But these three planes intersect in the diameter through A of the circumscribed sphere. Extend AO to meet the sphere again in E (Fig. 2). Then EA = diameter of circumsphere $= 2R$. Since A-BCD is a regular tetrahedron, AO is perpendicular to the base, BCD. Hence $\angle\ AOB$ is a right angle. Also $\angle\ EBA$, being inscribed in a semicircle, is a right angle. Hence in right $\triangle\ BOA$,

$$\overline{OA}^2 = \overline{AB}^2 - \overline{BO}^2 = a^2 - \frac{a^2}{3} = \frac{2a^2}{3},\ OA = \frac{a\sqrt{2}}{\sqrt{3}}.$$

$$\text{In right } \triangle EBA,\ \overline{BA}^2 = EA.\quad OA = 2R\cdot\frac{a\sqrt{2}}{\sqrt{3}}.$$

$$a^2 = 2R\cdot\frac{a\sqrt{2}}{\sqrt{3}},\qquad R = \frac{a\sqrt{3}}{2\sqrt{2}} = \frac{a\sqrt{6}}{4}.$$

Also solved by R. M. MATHEWS, NATHAN ALTSHILLER, A. M. HARDING, CLIFFORD N. MILLS, WALTER C. EELLS, A. H. HOLMES, HORACE OLSON, J. C. CLAGETT, J. W. CLAWSON.

CALCULUS.

361. Proposed by EMMA M. GIBSON, Drury College.

Determine the system of curves satisfying the differential equation

$$[(1 + x^2)^{1/2} + ny]dx + [(1 + y^2)^{1/2} + nx]dy = 0,$$

and show that the curve which passes through the point $(0, n)$ contains as part of itself the conic

$$x^2 + y^2 + 2xy(1 + n^2)^{1/2} = n^2.$$

(From Forsyth's *Differential Equations*, p. 41.)

SOLUTION BY GEO. W. HARTWELL, Hamline University.

The terms of the given differential equation may be arranged as follows:

$$(1 + x^2)^{\frac{1}{2}}dx + (1 + y^2)^{\frac{1}{2}}dy + nxdy + nydx = 0 \tag{1}$$

and the equation integrated immediately, giving

$$x\sqrt{1 + x^2} + y\sqrt{1 + y^2} + 2nxy + \log\,(x + \sqrt{1 + x^2})(y + \sqrt{1 + y^2}) = c. \tag{2}$$